Derivation of the (Absolute) Vorticity Equation

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As instructed, we start with the two components of the momentum equation, where Equation 1 and Equation 2 are the momentum terms, whereas the left side of these is the *material derivative* (i.e., partial time derivative and advection).

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_x \end{aligned} \tag{1}$$

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + fu + F_y \end{aligned} \tag{2}$$

For the further steps, the following identities come in useful:

$$(u, v, w) = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) \quad \text{(wind velocity components)} \tag{3}$$

$$\vec{U} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ and } \vec{V} = \begin{pmatrix} u \\ v \\ (0) \end{pmatrix} \quad \text{(wind vectors)} \tag{4}$$

$$\omega = \frac{\partial p}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial p}{\partial z} = w\frac{\partial p}{\partial z} = -w\rho g \quad \text{(omega)} \tag{5}$$

$$\frac{\partial p}{\partial z} = -\rho g \quad \text{(hydrostatic equation)} \tag{6}$$

$$\zeta_a = \hat{k} \cdot \boldsymbol{\nabla} \times \vec{U} + f = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \quad \text{(absolute vorticity)} \tag{7}$$

$$\vec{V}_g \cdot \boldsymbol{\nabla}\zeta_a = u\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial f}{\partial x}\right) + v\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial f}{\partial y}\right) \quad \text{(horizontal vorticity advection)} \tag{8}$$

$$\frac{\partial b}{\partial c}\frac{\partial a}{\partial b} = \frac{\partial a}{\partial c} \quad \text{(chain rule of differentiation (}*\text{))} \tag{9}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} \quad \text{(continuity equation)} \tag{10}$$

$\Longrightarrow$ $(*)$ about the chain rule: we use this here in a rather sloppy way. Strictly, only the chain rule $\frac{\mathrm{d}a(b(c))}{\mathrm{d}c} = \frac{\mathrm{d}a}{\mathrm{d}b}\frac{\mathrm{d}b}{\mathrm{d}c}$ with total derivatives is correct. When we write an atrocity such as Equation 16, the assumption is that $v(x(t), y(t), z(t))$ is a good approximation; we use the ∂ sign and pretend that $v(x, y, z, t)$ holds instead, with *"most"* of the time dependence coming through z. In any case, I hope that no mathematician or theoretical physicist will read this. The only reason to believe that it works is that we find the right equation in the end :^)

Furthermore, we assume that differentiation operators are commutative: $\frac{\partial^2 a}{\partial b \partial c} = \frac{\partial^2 a}{\partial c \partial b}$, which holds for "well-behaved" functions. The atmosphere doesn't have holes and is overall a rather smooth business, therefore this assumption seems reasonable.

The vorticity equation just describes the change of vorticity over time, $\frac{\partial \zeta_a}{\partial t}$. We can use Equation 7 for this. Note that this is the partial derivative; therefore, we use the $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ from Equation 1 and Equation 2, meaning we move all advection terms $i\frac{\partial a}{\partial i}$ to the right hand sides. Further note that $\frac{\partial f}{\partial t} = 0$.

$$\frac{\partial \zeta_a}{\partial t} = \frac{\partial}{\partial t}\left(\hat{k} \cdot \nabla \times \vec{U}\right) = \hat{k} \cdot \nabla \times \left(\frac{\partial}{\partial t}\vec{U}\right) \quad \text{(move derivative across curl, and hash out curl)} \tag{11}$$

$$= \frac{\partial}{\partial x}\left(-u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_y\right) \tag{12}$$

$$- \frac{\partial}{\partial y}\left(-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_x\right) \tag{13}$$

Now comes the first nasty part: writing out all terms individually. This is a brutal but simple way to later collect and group the individual terms back into a nice shape. This is mostly an application of the product rule, $\frac{\partial}{\partial x}(ab) = \frac{\partial a}{\partial x}b + a\frac{\partial b}{\partial x}$.

$$\begin{aligned}\frac{\partial \zeta_a}{\partial t} = &- \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - w\frac{\partial^2 v}{\partial x \partial z} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} - \frac{\partial f}{\partial x}u - f\frac{\partial u}{\partial x} + \frac{\partial F_y}{\partial x} \\ &+ \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial y \partial z} + \frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} - \frac{\partial f}{\partial y}v - f\frac{\partial v}{\partial y} - \frac{\partial F_x}{\partial y}.\end{aligned} \tag{14}$$

Phew! These terms are all we need. Now we just need to arrange them nicely. Spoiler alert! These are the individual named terms from the vorticity equation we're trying to derive: Horizontal Vorticity Advection, Vertical Vorticity Advection, Tilting, Stretching, and Friction.

Note that the Pressure Gradient terms vanish: due to interchangeable derivatives and differing signs, these terms add up to 0; we also assumed that the air's density doesn't vary horizontally. Also, we can identify the friction terms $\frac{\partial F_y}{\partial x}$ and $\frac{\partial F_x}{\partial y}$ right away; we cannot further modify them, so they will stay like that.

As a next step, we can identify six terms from Horizontal Vorticity Advection:

$$\textit{Equation 8} = u\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial f}{\partial x}\right) + v\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial f}{\partial y}\right) = \vec{V}_g \cdot \nabla \zeta_a \tag{15}$$

Next, we can identify the vertical vorticity advection (VVA), and rewrite it:

$$w\frac{\partial^2 v}{\partial x \partial z} = \frac{\partial z}{\partial t}\frac{\partial^2 v}{\partial x \partial z} \quad \text{(notice chain rule)} = \frac{\partial^2 v}{\partial t \partial x} \tag{16}$$

$$w\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial t \partial y} \tag{17}$$

Let us exploit the chain rule in an unintuitive way again, and insert a differentiation w.r.t. pressure:

$$\frac{\partial^2 v}{\partial t \partial x} = \frac{\partial p}{\partial t}\frac{\partial^2 v}{\partial p \partial x} = \omega\frac{\partial^2 v}{\partial p \partial x} \tag{18}$$

$$\frac{\partial^2 u}{\partial t \partial y} = \frac{\partial p}{\partial t}\frac{\partial^2 u}{\partial p \partial y} = \text{(analogous)} \tag{19}$$

$$\tag{20}$$

By subtracting Equation 18 and Equation 19, we find

$$\omega\frac{\partial^2 v}{\partial p \partial x} - \omega\frac{\partial^2 u}{\partial p \partial y} + \underbrace{\frac{\partial f}{\partial t}}_{=0(*)} = \omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \omega\frac{\partial \zeta_a}{\partial p} \tag{21}$$

Note for (∗) that the time derivative of the Coriolis factor vanishes, thus we can indeed write this as time derivative of the *absolute* vorticity.

The Tilting term also needs a somewhat non-obvious application of the chain rule, and using Equation 6:

$$\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} = -\frac{1}{\rho g}\frac{\partial \omega}{\partial x}\underbrace{\frac{\partial p}{\partial z}}_{=-\rho g}\frac{\partial v}{\partial p} = \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} \tag{22}$$

$$\text{and analogously } \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} = \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p} \tag{23}$$

Now for the Stretching: these terms look quite chaotic, but there is a system; they are the product of ζ_a with $\nabla\vec{V}$:

$$-\zeta_a \nabla \vec{V}_g = -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{24}$$

$$= -\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - f\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - f\frac{\partial v}{\partial y} \tag{25}$$

Yay – another six terms accounted for! Now, how to write this elegantly? From Equation 10 we know that

$$\frac{\partial \omega}{\partial p} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\boldsymbol{\nabla}\vec{V}_g,$$

therefore replace this term in Equation 24:

$$\textit{Equation 25} = \zeta_a \frac{\partial \omega}{\partial p} \tag{26}$$

Aaaand – we're already done. Let us assemble the results. We only need to be careful about the signs that the individual terms appear with in Equation 14:

$$\frac{\partial \zeta_a}{\partial t} = -\vec{V}_g \cdot \boldsymbol{\nabla}\zeta_a \quad \text{(Equation 15)} \tag{27}$$

$$-\omega\frac{\partial \zeta_a}{\partial p} \quad \text{(Equation 21)} \tag{28}$$

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right) \quad \text{(Equation 23)} \tag{29}$$

$$+\zeta_a\frac{\partial \omega}{\partial p} \quad \text{(Equation 26)} \tag{30}$$

$$+\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \quad \text{(friction)}, \tag{31}$$

and in its full beauty:

$$\frac{\partial \zeta_a}{\partial t} = -\vec{V}_g \cdot \boldsymbol{\nabla}\zeta_a - \omega\frac{\partial \zeta_a}{\partial p} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right) + \zeta_a\frac{\partial \omega}{\partial p} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \quad \text{(vorticity equation)} \tag{32}$$

Divergence Equation

To derive the Divergence Equation (DE) – which describes "the rate of change of horizontal divergence on a parcel"[1] –, we can take the divergence of the vectorial equation of motion, which we know from the previous part:

[1] https://glossary.ametsoc.org/wiki/Divergence_equation

$$\frac{D}{Dt}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}-\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_x\\ -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_y\end{pmatrix} = \tag{33}$$

although we will neglect friction for now (because we discuss air parcels aloft): $F_i = 0$ for $i = x, y$. The derivation of the DE is possible by taking – unsurprisingly – the divergence of Equation 33:

$$\begin{aligned}\nabla_{x,y}\cdot\frac{D}{Dt}\begin{pmatrix}u\\v\end{pmatrix} &= \frac{\partial}{\partial x}\frac{Du}{Dt} + \frac{\partial}{\partial y}\frac{Dv}{Dt}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\end{aligned} \tag{34}$$

Note that for the version of the DE that we're trying to derive, we set $\nabla = \nabla_{x,y}$, i.e. the horizontal derivative, as was given away in the very first sentence above.

Now we expand the material derivative

$$\frac{DX}{Dt} = \left(\frac{\partial X}{\partial t} + u\frac{\partial X}{\partial x} + v\frac{\partial X}{\partial y}\right):$$

$$\begin{aligned}\nabla_{x,y}\cdot\frac{D}{Dt}\begin{pmatrix}u\\v\end{pmatrix} =& \frac{\partial^2 u}{\partial x\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\ v\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial x\partial z}\\ &+ \frac{\partial^2 v}{\partial y\partial t} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y\partial x} + \left(\frac{\partial v}{\partial y}\right)^2\ v\frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial y\partial z} \quad \text{(LHS)}\end{aligned} \tag{35}$$

$$= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + f\frac{\partial v}{\partial x} - \frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} - \frac{\partial f}{\partial y}u - f\frac{\partial u}{\partial y} \quad \text{(RHS from Equation 33)} \tag{36}$$

Now we shall rearrange the heap of terms into a useful form. The *new* LHS, which we will guess, will be the material derivative of the horizontal wind divergence:

$$\frac{D}{Dt}\left(\nabla_{x,y}\cdot\vec{U}\right) = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{37}$$

And the green color already gives away which terms will be on the new LHS.

On the *new* RHS, we first collect the red terms and rearrange them as result of a quadratic expansion:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} \quad \text{(for new RHS)} \tag{38}$$

And while we're at it, we can add the purple terms, too:

$$\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{39}$$

Now only the violet terms need to be handled; as in the vorticity equation's derivation, a bit of sloppy chain rule application gets us to an expression in terms of ω:

$$\frac{\partial^2 u}{\partial x\partial t} = \frac{\partial}{\partial x}\frac{\partial p}{\partial t}\frac{\partial u}{\partial p} = \frac{\partial\omega}{\partial x}\frac{\partial u}{\partial p} \tag{40}$$

$$\frac{\partial^2 v}{\partial y\partial t} = \frac{\partial}{\partial y}\frac{\partial p}{\partial t}\frac{\partial v}{\partial p} = \frac{\partial\omega}{\partial y}\frac{\partial v}{\partial p} \tag{41}$$

The new right hand side (RHS) comes from Equation 33 (note that $\frac{\partial f}{\partial x} = 0$):

$$\nabla_{x,y} \cdot \frac{D}{Dt}\begin{pmatrix} u \\ v \end{pmatrix} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + f\frac{\partial v}{\partial x} - \frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} - \frac{\partial f}{\partial y}u - f\frac{\partial u}{\partial y} \tag{42}$$

These terms can also be written more nicely: The pressure Laplacian term can be written in terms of the geopotential, because $\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$ (eq. (2.16) in the textbook, or just the ideal gas law), so that

$$\begin{aligned} \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} &= \nabla^2 \Phi \\ &= \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \underbrace{\frac{\partial^2 \Phi}{\partial z^2}}_{=0} \right). \end{aligned} \tag{43}$$

The remaining terms can be left as-is. Now we need to arrange the terms on the new LHS and RHS, adjusting signs. The green terms stay in place, so that Equation 35 becomes the new LHS (as promised), and everything else goes on the new RHS:

$$\frac{D}{Dt}\left(\nabla_{x,y} \cdot \vec{U}\right) \tag{44}$$

$$= -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) - \left(\frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p} + \frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p}\right) + \frac{\partial f}{\partial y}u - \left(f\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \nabla^2 \Phi \tag{45}$$

$$= -\left(\nabla_{x,y} \cdot \vec{U}\right) + 2\frac{\partial(u, v)}{\partial(x, y)} - \boldsymbol{\nabla}\omega \cdot \frac{\partial \vec{U}}{\partial p} - \beta u + f\zeta + \nabla^2 \Phi \tag{46}$$

The final Equation 46 can be found in references, such as `https://glossary.ametsoc.org/wiki/Divergence_equation`.

QG Vorticity equation

(This was task 2.)

The QG vorticity equation takes in some simplifications compared to the full vorticity equation, and is somewhat easier to derive.

It all begins with the QG momentum equations,

$$\frac{\mathrm{d}u_g}{\mathrm{d}t} = f_0 v_{ag} + \beta y v_g = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \tag{47}$$

$$\frac{\mathrm{d}v_g}{\mathrm{d}t} = -f_0 u_{ag} - \beta y u_g = \ldots \tag{48}$$

The goal is an expression for

$$\frac{\partial \zeta_g}{\partial t} = \frac{\partial}{\partial t}\boldsymbol{\nabla} \times \begin{pmatrix} u_g \\ v_g \end{pmatrix} = \boldsymbol{\nabla} \times \left(\frac{\partial}{\partial t}\begin{pmatrix} u_g \\ v_g \end{pmatrix}\right) \tag{49}$$

that is, the change over time of geostrophic vorticity (analogous to the full vorticity). As we mostly care about the z component of vorticity, we deal with

$$\frac{\partial \zeta_g}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{50}$$

and remember that

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + f_0 v_{ag} + \beta y v_g$$

(and analogous for v) due to the material derivative in Equation 47.

Thus,

$$\frac{\partial \zeta_g}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{51}$$

$$= \frac{\partial}{\partial x}\left(-u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - f_0 u_{ag} - \beta y u_g\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + f_0 v_{ag} - \beta y v_g\right) \tag{52}$$

$$= -\frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x} - u_g\frac{\partial^2 v_g}{\partial x^2} - \frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y} - v_g\frac{\partial^2 v_g}{\partial x \partial y} - f_0\frac{\partial u_{ag}}{\partial x} - \beta y\frac{\partial u_g}{\partial x}$$

$$+ \frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x} + u_g\frac{\partial^2 u_g}{\partial y \partial x} + \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y} + v_g\frac{\partial^2 u_g}{\partial y^2} - f_0\frac{\partial v_{ag}}{\partial y} - \beta v_g - \beta y\frac{\partial v_g}{\partial y} \tag{53}$$

And yet again, we need to process term by term, assembling them into a nicer (vectorial) form. First, the blue terms can – with the help of the mass continuity equation – be written as

$$-f_0\left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y}\right) = +f_0\frac{\partial \omega}{\partial p} \tag{54}$$

The orange terms can be re-sorted into

$$-u_g\frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) \tag{55}$$

$$-v_g\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \underbrace{f}_{\frac{\partial f}{\partial y}=\beta!}\right) \tag{56}$$

which can be written very tersely as

$$(\textit{Equation 55} + \textit{Equation 56}) = -\vec{V}_g \cdot \nabla(\zeta_g + f) \tag{57}$$

wherein we just substitute definitions for ζ_g, and notice that $\frac{\partial f}{\partial x} = 0$.

Here, we already arrive at the QG vorticity equation, as given in the Synoptics textbook:

$$\frac{\partial \zeta_g}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{58}$$

$$= -\vec{V}_g \cdot \nabla(\zeta_g + f) + f_0\frac{\partial \omega}{\partial p} \qquad \text{(QG Vorticity equation.)} \tag{59}$$

Now we need to sort out the remaining black and violet terms from Equation 53. The black terms are approximating meridional changes of the Coriolis force, which are relatively small when multiplied with the meridional change of winds; we therefore neglect them.

The violet terms turn out to be the relative-vorticity part of the stretching term known from the full vorticity Equation 32 (there: $(\zeta + f)\frac{\partial \omega}{\partial p}$):

$$\textit{Equation } 53 \text{ contains } -\left(\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y}\right)\left(\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial x}\right)=-\zeta_g(\boldsymbol{\nabla}\cdot\vec{V}_g)=\zeta_g\frac{\partial\omega}{\partial p} \tag{60}$$

After some research in the Synoptics textbook, I have been convinced that one part of the QG approximation involves simply ignoring this term: "*In the full equation, the magnitude of the stretching term is proportional to the vorticity itself, meaning that zones of large preexisting vorticity are sites for preferential vorticity growth, and also that an exponential feed- back takes place. In the QG version, this effect is absent.*" (p. 41)

With this, all terms are again accounted for, and we obtained the *QG vorticity Equation 59*.